Incomparable!

Carlos Alós-Ferrer and Paulo Natenzon[*]

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Abstract

Incomparability between alternatives challenges economic models with complete preferences, yet remains inadequately addressed in stochastic choice theory. We model incomparability as the limiting case of infinite differentiation between alternatives. Our generalization of the moderate utility model distinguishes merely difficult comparisons from truly incomparable options, yields a characterization of binary choice probabilities using two simple postulates, and reveals an endogenous partition of alternatives into "galaxies" of comparable options.

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[*] First version posted April 2025. Comments Welcome. Carlos Alós-Ferrer, Lancaster University Management School, `c.alosferrer@lancaster.ac.uk`. Paulo Natenzon, Washington University in Saint Louis, `pnatenzon@wustl.edu`.

1 Introduction

Completeness of preferences remains one of the most persistent yet contestable axioms in microeconomic theory. While the traditional rational economic agent employed in the bulk of modern economic analysis is always able to compare any two alternatives —be they consumptions baskets, health insurance plans, carreer choices, and so on— at least as early as Aumann (1962) the literature has recognized that, in may situations, it is entirely plausible that a person will be incapable or unwilling to compare some options. In particular, she may be unwilling to declare that an option A is at least as good as another option B and also unwilling to declare the reverse statement. The implication of this observation is that, both descriptively and normatively, microeconomic models of decision making ought to allow for the possibility that two options A and B are simply *incomparable*.

Virtually all existing models of incomparability start from a binary relation to describe an incomplete preference and obtain a representation using some form of multi-criterion decision making (see e.g. Aumann (1962), Bewley (1986, 2002), Ok (2002), Dubra, Maccheroni and Ok (2004), Eliaz and Ok (2006), Ok, Ortoleva and Riella (2012), Galaabaatar and Karni (2013), Faro (2015)). In these models, the decision maker is able to rank options when the multiple criteria employed agree, but declares options incomparable when there is any disagreement among the criteria. For example, in Bewley (1986, 2002) the individual has a single utility function but holds a set of different beliefs about the likelihood of each state of nature. Each belief in this set could be the opinion of a different expert, for instance. When comparing two uncertain prospects, the decision maker declares one better when it yields higher expected utility according to all the experts; if there is any disagreement among experts, the options are declared incomparable.

In this paper, we propose a new approach to model incomparability, more directly tied to observable choice behavior. In our framework, the analyst observes $\rho(x, y)$, the probability that x is chosen over y in a binary comparison, and an incomplete preference is revealed by the observed binary stochastic choices. An advantage of this framework is that it allows us to distinguish true incomparability in the normative sense from mere indecisiveness in reaching a difficult but feasible decision.

Formally, we generalize the moderate utility model of binary stochastic choice characterized in He and Natenzon (2024). The framework represents choice probabilities as a function of utility differences normalized by a distance metric that captures the comparability of alternatives. We generalize this framework by allowing the distance metric to take infinite values while preserving its mathematical properties. When the distance between two alternatives is infinite, we interpret this as fundamental incomparability —the decision maker can make no meaningful comparison and therefore chooses randomly with equal probability. Notably, by embedding incomparability within a stochastic choice framework, we are able to provide a direct link between the cognitive structure of incomparability and empirically testable choice probabilities.

The structure imposed by our axioms induces an endogenous partition of the universe of alternatives into "galaxies" of comparable options. Within each galaxy, the original moderate utility model applies, with choice probabilities determined by utility differences normalized by finite distances. Between galaxies, alternatives are incomparable, and choice is governed by pure randomness. This partition emerges naturally from the behavioral properties of the model rather than being imposed exogenously, providing a principled account of how incomparability structures emerge from choice behavior.

We also address an important question of identification: upon observing that x and y are equally likely to be chosen, how can we tell if the decision maker is indifferent or finds them truly incomparable? The philosopher Raz (1985) anticipates our solution: find a third option z that is comparable to x. For example, z could be x enhanced by a small monetary bonus. As long as the decision maker likes money, z should be clearly better than x. Now if the choice between z and y is still fifty-fifty, then the original options are incomparable; while if z is chosen more often than y, the original options must have been indifferent. Using this intuition, we show that a simple data richness condition guarantees the partition of alternatives into galaxies of comparable options is uniquely pinned down by observed choice behavior.

The remainder of the paper is organized as follows. Section 2 introduces the generalized moderate utility model. Section 3 shows that, under a weak richness condition, incomparability is uniquely identified from choice. Section 4 presents our behavioral characterization based on two simple postulates. Section 7 relates our results to the existing literature. Section 8 concludes.

1.1 Related Literature

1.1.1 Experimental Evidence of Preference Incompleteness

The empirical measurement of preference incompleteness presents unique methodological challenges, as incompleteness is inherently difficult to observe directly through standard choice data. The pioneering experimental work by Danan and Ziegelmeyer (2006) addressed this challenge by developing an innovative design that allowed subjects to postpone decisions at a small cost, thereby revealing potential incompleteness in their preferences. Their findings provided the first direct experimental evidence that preferences over risky prospects are significantly incomplete for many decision-makers. Recent work by Nielsen and Rigotti (2024) developed a more direct methodology for eliciting incomplete preferences over monetary gambles with subjective uncertainty, finding that about 40% of subjects explicitly express incompleteness, with evidence suggesting that up to 98% exhibit indirect signs of incompleteness. Halevy et al. (2023) proposed an incentive-compatible mechanism that bounds the behavior rationalizable by complete preferences, finding that choices incompatible with Subjective Expected Utility are usually also incompatible with general models of complete preferences. Several studies have linked incompleteness to observable economic behaviors: Cettolin and Riedl (2019) demonstrated that individuals with incomplete preferences may prefer to randomize their choices, while Cubitt et al. (2015) showed that subjects often report low confidence in decisions that involve potentially incomplete preferences. Recent work by Chambers et al. (2022) has developed methods to empirically determine preference incompleteness through willingness-to-pay measures, providing economists with additional tools to identify incomparability in laboratory settings. Collectively, these experiments confirm that preference incompleteness is not merely a theoretical curiosity but a significant empirical phenomenon with important implications for economic behavior.

2 Model

Primitives

Let X be a non-empty finite set of alternatives. We call $\rho : X \times X \to [0,1]$ a binary random choice rule on X whenever it satisfies $\rho(x,y) + \rho(y,x) = 1$ for all $x, y \in X$. Here, $\rho(x,y)$ measures the probability that a decision maker selects x when presented with a binary comparison between x and y. In addition to the individual stochastic choice interpretation, ρ can also be given a population interpretation, where $\rho(x,y)$ is the probability that a randomly selected consumer from a heterogeneous population of agents chooses x over y.

The original Moderate Utility Model

The moderate utility model (Halff, 1976; He and Natenzon, 2024) provides a framework for binary stochastic choice that bridges utility-based and similarity-based approaches to decision theory. The model posits that choice probabilities are determined by the interplay between utility differences and option comparability. Formally, the moderate utility model represents $\rho(x,y)$ as:

$$\rho(x,y) = F\left(\frac{u(x)-u(y)}{d(x,y)}\right) \tag{1}$$

Where:

- $u : X \to \mathbb{R}$ is a utility function capturing the value of each option
- $d : X \times X \to \mathbb{R}_+$ is a distance metric satisfying:
  - Non-negativity: $d(x,y) \geq 0$ for all $x, y \in X$
  - Identity of indiscernibles: $d(x,y) = 0$ if and only if $x = y$
  - Symmetry: $d(x,y) = d(y,x)$ for all $x, y \in X$
  - Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$
- $F : \mathbb{R} \to [0,1]$ is a strictly increasing cumulative distribution function with $F(t) + F(-t) = 1$ and, in particular $F(0) = 1/2$.

The key insight of this model is that the probability of choosing x over y depends not just on the utility difference $u(x) - u(y)$, but on this difference normalized by the distance between the options. The distance metric $d(x,y)$ captures how differentiated or comparable the options are. For highly differentiated options (large distance), even substantial utility differences produce choice probabilities closer to $1/2$, reflecting greater choice stochasticity under difficult comparisons.

The moderate utility model successfully unifies several important choice models used in economics, including Hotelling's (1929) and Salop's (1979) spatial competition models, nested logit (McFadden, 1978), covariance probit (Bock and Jones, 1958), and elimination-by-aspects (Restle, 1961; Tversky, 1972). Next, we propose a generalization of this model to capture fundamental incomparability between options.

The Generalized Moderate Utility Model with Incomparability

While the moderate utility model elegantly captures degrees of comparability through the distance metric, it implicitly assumes all alternatives are comparable to some extent. However, philosophers have long recognized that some alternatives may be fundamentally incomparable (Raz, 1986; Chang, 1997). We now extend the moderate utility model to incorporate genuine incomparability while preserving its simple formula and tractability.

We introduce an extended distance metric $d : X \times X \to [0, \infty]$ that maps to the extended non-negative real line, including infinity. The interpretation is natural: infinite distance represents absolute incomparability between alternatives. The generalized moderate utility model defines choice probabilities according to equation (1), substituting an extended an extended metric d for the usual standard metric.

This formulation maintains all properties of a distance metric, with arithmetic involving infinity defined conventionally: for any finite $t \in \mathbb{R}$, $t + \infty = \infty + t = \infty$ and $\infty + \infty = \infty$. For any finite t, we have $t/\infty = 0$.

Under incomparability, that is, with $d(x, y) = \infty$, the choice probability given by equation (1) becomes exactly 1/2, representing pure randomness in choice—the decision maker cannot rationally discriminate between incomparable alternatives.

Figure 1 plots the choice probabilities in the generalized moderate utility model as a function of the difference in value between two options x and y, and for three different levels of differentiation $d(x, y)$ between the options. The standard model with positive finite levels of differentiation yields a sigmoid graph, in line with the stochastic choice models typically employed in discrete choice estimation. Taking the limit as differentiation goes to zero yields a step function and corresponds to deterministic utility maximization. Conversely, taking the limit as differentiation goes to infinity yields a perfectly horizontal one half probability line, reflecting a complete inability to compare x and y.

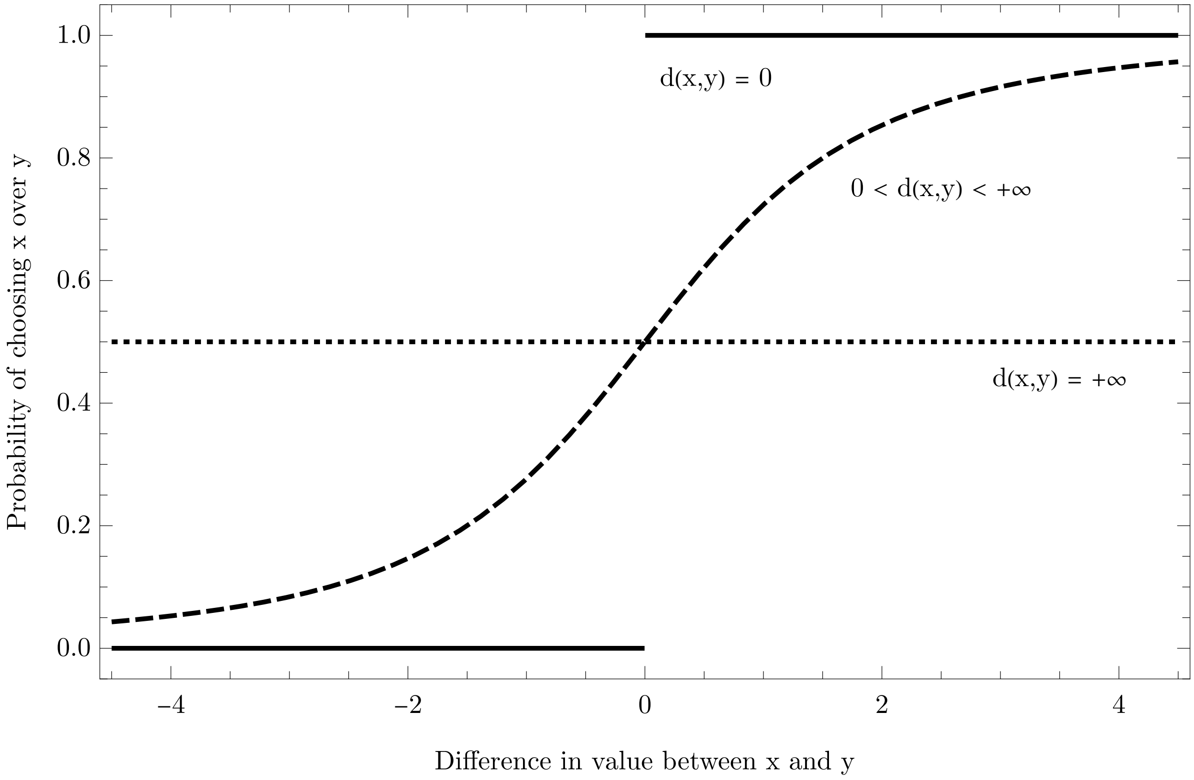

Figure 1: Choice probabilities in the moderate utility model as function of the difference in value between two options x and y, and for three different levels of differentiation $d(x, y)$ between the options. The standard model with positive finite levels of differentiation yield a sigmoid in line with soft-maximization and a large family existing stochastic choice models used in discrete choice estimation. Taking the limit as differentiation goes to zero yields a step function and corresponds to deterministic utility maximization. Conversely, taking the limit as differentiation goes to infinity yields a perfectly horizontal one-half probability line, reflecting a complete inability to compare x and y.

3 Identification

Our model generates 50-50 choice between two options due to indifference (when two options have the same utility), but it also produces 50-50 choice due to incomparability (when the differentiation between the options is infinite). Upon observing or hypothesizing that $\rho(x, y) = 1/2$, how can an external observer assess if x and y are incomparable or comparable but indifferent? The philosopher Joseph Raz (1985), discussing incomparability (which he interchangeably calls incomensurability) anticipates the solution:

> We have here a simple way of determining whether two options are incommensurate given that it is known that neither is better than the other. If it is possible for one of them to be improved without thereby becoming better than the other or if there can be another option which is better than the one but not better than the other then the two original options are incommensurate. [p. 121]

Such a solution is, of course, not necessarily always available. For example, if the universe of alternatives contains three distinct options and $\rho(x, y) = \rho(y, z) = \rho(x, z) = 1/2$ it is impossible to know which pairs are indifferent and which pairs are incomparable. In particular, this ρ admits multiple generalized moderate utility representations, where the distance given by d can be either finite or infinite for each pair. However, a simple data richness condition is sufficient to guarantee full identification of incomparability:

Proposition 1. *Suppose that for every $x \in X$ there exists $y \in X$ such that $\rho(x, y) \neq 1/2$. Then whenever ρ admits a generalized moderate utility representation, the partition of X into galaxies of comparable options is unique.*

A natural way to satisfy this richness condition is to pair options with small monetary bonuses: assuming that a small monetary bonus increases the value of an option without affecting its comparability, and that the decision maker clearly prefers to get the bonus, the conditions of Proposition 1 are easily satisfied.

4 Behavioral characterization

In this section we prove that our model is characterized by two simple postulates on binary choice. A choice rule ρ satisfies *moderate transitivity* when for all $i, j, k \in Z$,

$$\min\{\rho(i,j), \rho(j,k)\} \geq 1/2 \text{ implies } \begin{cases} \rho(i,k) > \min\{\rho(i,j), \rho(j,k)\} \\ \text{or} \\ \rho(i,k) = \rho(i,j) = \rho(j,k) \end{cases}$$

We weaken moderate transitivity by requiring a strict inequality in the hypothesis:

Axiom 1: $\min\{\rho(i,j), \rho(j,k)\} > 1/2 \text{ implies } \begin{cases} \rho(i,k) > \min\{\rho(i,j), \rho(j,k)\} \\ \text{or} \\ \rho(i,k) = \rho(i,j) = \rho(j,k) \end{cases}$

Axiom 1, in turn, is stronger than Partial Stochastic Transitivity (PST) studied in Fishburn (1973):

PST: $\min\{\rho(i,j)\, \rho(j,k)\} > 1/2$ implies $\rho(i,k) \geq \min\{\rho(i,j), \rho(j,k)\}$

It is easy to verify directly from the definitions that Moderate Transitivity ⇒ Axiom 1 ⇒ PST. We further discuss the empirical gap between the three axioms in Section **??**.

The second consequence of allowing for infinite differentiation is a partition of the space of options into galaxies of comparable alternatives. The behavioral consequence of this partition is described by the following axiom:

Axiom 2: If $\rho(x,y) \neq 1/2 \neq \rho(z,w)$ and $\rho(x,z) = 1/2 = \rho(y,z)$, then $\rho(x,w) = 1/2$.

Interpreting Axiom 2 is straightforward: the hypothesis $\rho(x,y) \neq 1/2 \neq \rho(z,w)$ reveals that x, y must be in the same galaxy of comparable options; and likewise z, w must belong to the same galaxy. However since z is chosen 50-50 against both x and y, those two galaxies must be separate. Hence, w must be incomparable to x and y. See Figure 2 for an illustration.

We are now ready to state the main characterization result:

Theorem 1. *A choice rule ρ is a generalized MUM iff it satisfies Axioms 1 and 2.*

The proof can be found in the Appendix. The first step is to show that if binary choice satisfies Axiom 1, then the universe of options X is endogenously partitioned into groups of options, which we call "galaxies", such that the comparisons restricted to each galaxies satisfy the moderate transitivity postulate, and comparisons across galaxies are always purely random 50-50 choice.

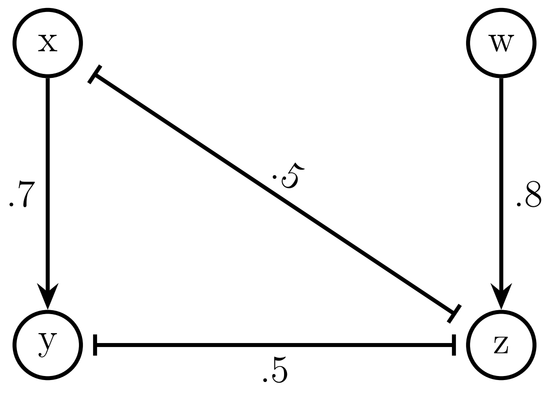

Figure 2: Illustrates how Axiom 2 captures the partition of the universe of alternatives into galaxies of comparable options. Options x and y must belong to the same galaxy since $\rho(x, y) = 7$. Option z must belong to a different galaxy since $\rho(x, z) = 1/2 = \rho(y, z)$. Hence any option w with $\rho(w, z) \neq 1/2$ must be in z's galaxy, and therefore incomparable to x and y. Thus Axiom 2 requires that $\rho(w, x) = \rho(w, y) = 1/2$.

5 Revealed Incomplete Preference

Given a choice rule ρ on X define the revealed preference relation $\succsim_\rho \subset X \times X$ by

$$x \succsim_\rho y \ \text{ iff } \begin{cases} \rho(x, y) > 1/2 \\ \text{or} \\ \rho(x, y) = 1/2 \text{ and } \rho(x, z) \neq 1/2 \neq \rho(y, z) \text{ for some } z \in X \end{cases}$$

There are two kinds of incomplete preference relations studied in the litearture. The most common is represented by multiple criteria: (

Multi-criterion preference: $\succsim = \bigcap_i \succsim_i$ where each $\succsim_i$ complete and transitive

Another type of incomplete preference arises from interval order representations:

Interval order: $u, v : X \to \mathbb{R}$, $u \leq v$ and $x \succ y$ iff $u(x) > v(y)$.

Proposition 2. *If ρ is a generalized MUM, the revealed preference $\succsim_\rho$ is a multi-criterion preference but not necessarily an interval order preference.*

6 Finite data

While Proposition 1 shows that incomparability and indifference can be uniquely separated within the model, we now note that the issue is more subtle when we talk about inference from finite choice data in empirical aplications of the theoretical framework.

Proposition 3. *Every generalized moderate utility model ρ can be arbitrarily well approximated by a moderate utility model.*

Proof. Let ρ be a moderate utility model on the non-empty finite set X, and let d be its extended distance metric. For each $n \in \mathbb{N}$ let d_n be the function given by

$$d_n(x, y) = \begin{cases} d(x, y), & \text{if } d(x, y) < \infty \\ n, & \text{if } d(x, y) = \infty \end{cases}$$

and note that for all n large enough, n is larger than any intra-galaxy distance in the original d and therefore d_n will satisfy the triangle inequality. Since X is finite, ρ is can be mapped to a point in a finite dimensional euclidean space. As n goes to infinity, the moderate utility model generated by substituting the finite d_n for d will be arbitrarily close to the original ρ. $\square$

7 Relation to existing models

Figure 3 situates our generalized moderate utility model within the broader landscape of binary stochastic choice theory. The figure illustrates the hierarchical relationships between observable properties (left column) and their corresponding theoretical models (right column), organized from the weakest constraints at the top to the strongest at the bottom.

At the most general level, weak transitivity corresponds to weak utility representations, which simply require options to be assigned scalar utilities that generate choice probabilities above 1/2 when the utility of one option exceeds another. As we move down the hierarchy, we encounter increasingly restrictive models with stronger empirical implications.

The dotted box highlights the region where our contribution fits. Fishburn (1973) established that partial stochastic transitivity corresponds to strict partial order models. Below this, our Theorem 1 demonstrates that Axioms 1 and 2 —which systematically extend moderate transitivity to accommodate incomparability— is equivalent to our proposed framework. This represents a natural extension of the fundamental result in He and Natenzon (2024) that moderate transitivity characterizes the moderate utility model.

Further down the hierarchy, we see more restrictive models. Tversky and Russo (1969) showed that strong stochastic transitivity combined with positivity (i.e. that choice probabilities are strictly positive for every option) characterizes simple scalability models, where choice probabilities depend solely on the difference in utilities. Fudenberg et al. (2015) established that acyclicity with positivity corresponds to Fechnerian utility models, where choice probabilities are a strictly increasing function of utility difference. At the most restrictive level, Luce (1959) proved that the product rule with positivity characterizes logit models, where choice probabilities follow a specific functional form involving exponential utilities.

This hierarchical structure clarifies how our generalized MUM contributes to the literature. Unlike weaker models that permit too many patterns of behavior to be empirically useful, or stronger models that impose unrealistic constraints on behavior, the generalized moderate utility model provides a principled way to model incomparability while preserving the predictive power that makes the moderate utility model valuable in applications ranging from industrial organization to behavioral economics.

8 Conclusion

The generalized moderate utility model offers a mathematically rigorous framework for modeling choice under incomparability. By allowing infinite distances between alternatives, we capture the intuition that some options defy rational comparison while preserving the behavior of comparable options within the established moderate utility framework. This extension provides fertile ground for new empirical investigations into the boundaries of rational comparison and the nature of stochastic choice when faced with truly incomparable alternatives.

A Proof of Theorem 1

Define the binary relation $\sim$ on X by letting $x \sim x$ for every x, and, for each pair $x \neq y$, let $x \sim y$ if and only if the following holds:

(i) $\rho(x, y) \neq 1/2$; or

(ii) there is $z \in X$ such that $\rho(x, z) \neq 1/2$ and $\rho(y, z) \neq 1/2$.

Lemma 1. *Under Axiom 2, the binary relation $\sim$ is an equivalence relation.*

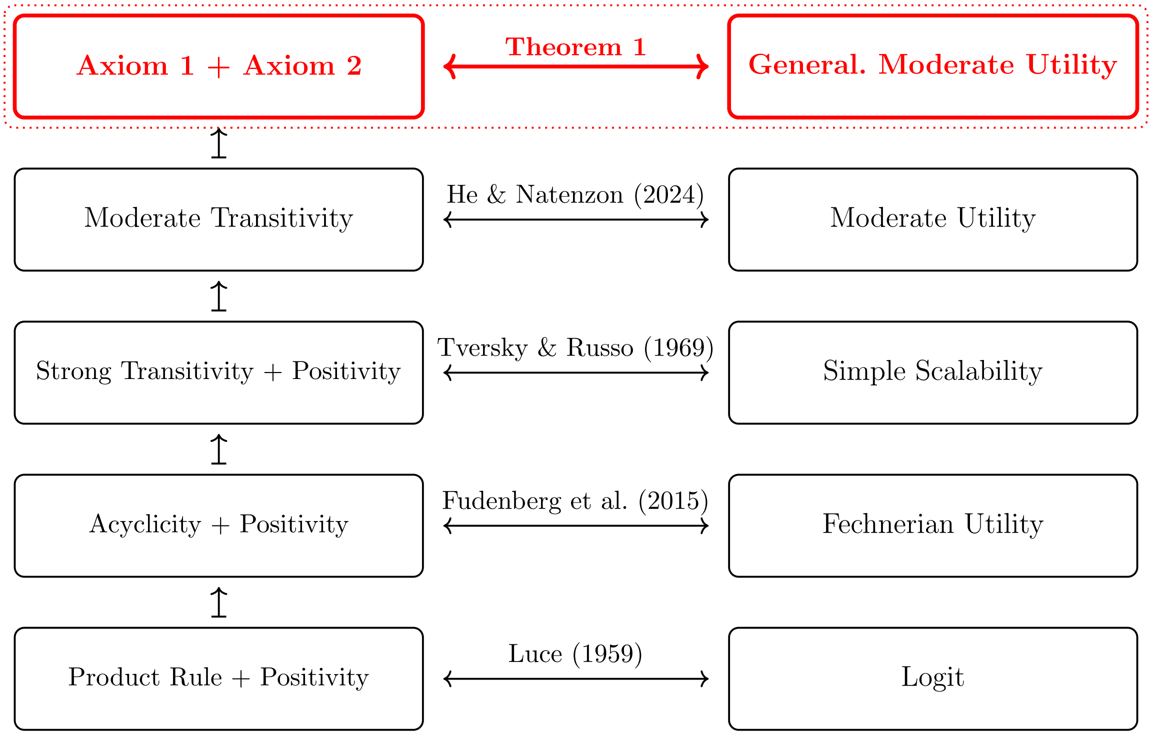

Figure 3: Relationship between models and postulates on choice probabilities for binary stochastic choice over a finite set of options. A double arrow (↔) indicates equivalence while an arrow (↦) indicates implication in the direction of the arrow and failure of implication in the opposite direction.

Proof. The relation $\sim$ is reflexive and symmetric by construction. It remains to show that $\sim$ is transitive. Let $x \sim y$ and $y \sim z$ for three distinct options $x, y, z \in X$. We show that $x \sim z$ must hold. When $\rho(x, z) \neq 1/2$ we directly obtain $x \sim z$ from part (i) of the definition. It remains to show $x \sim z$ when $\rho(x, z) = 1/2$.

Case 1: $\rho(x, y) \neq 1/2$. Note that in this case we must have $\rho(y, z) \neq 1/2$, for if we had $\rho(y, z) = 1/2$ with $y \sim z$, there would exist $w \in X$ such that $\rho(y, w) \neq 1/2 \neq \rho(z, w)$. By Axiom 2, we would have $\rho(x, w) = 1/2$. Applying Axiom 2 again to $\rho(y, x) \neq 1/2 \neq \rho(z, w)$ and $\rho(y, z) = 1/2 = \rho(x, z)$ would yield $\rho(y, w) = 1/2$, a contradiction. Hence $\rho(x, y) \neq 1/2 \neq \rho(y, z)$, and part (ii) of the definition yields $x \sim z$.

Case 2: $\rho(x, y) = 1/2$. By an argument symmetric to Case 1, we must have $\rho(y, z) = 1/2$. By the definition of the relation $\sim$, there must exist $w_1, w_2 \in X$ such that $\rho(x, w_1) \neq 1/2 \neq (y, w_1)$ and $\rho(y, w_2) \neq 1/2 \neq (z, w_2)$. If $w_1 = w_2$ then by (ii) we have $x \sim z$ as desired. So suppose $w_1 \neq w_2$. If we had $\rho(z, w_1) = 1/2$ then applying Axiom 2 to w_1, y, z, w_2 we would obtain $\rho(y, w_2) = 1/2$, a contradiction. So it must be $\rho(z, w_1) \neq 1/2$ and since $\rho(x, w_1) \neq 1/2$ part (ii) yields $x \sim z$ as desired. This completes the proof that the relation $\sim$ is transitive, and therefore an equivalence relation. $\square$

Since X is finite, the equivalence relation $\sim$ induces a finite partition

$$X = X_1 \cup X_2 \cup \cdots \cup X_k$$

For the rest of the proof we refer to X as the *universe* of alternatives and to each X_i of the partition as a *galaxy*.

Lemma 2. *ρ is moderately transitive when restricted to each X_i.*

Proof. Let $x, y, z \in X_i$ and suppose $\rho(x, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$. If $\rho(x, y) > 1/2$ and $\rho(y, z) > 1/2$ then Axiom 1 yields the desired conclusion.

If $\rho(x, y) = 1/2$ and $\rho(y, z) = 1/2$, since $x \sim y$ there exists a $w \in X$ such that $\rho(x, w) \neq 1/2 \neq \rho(y, w)$. If we had $\rho(x, z) \neq 1/2$ then x, y, z, w_1 would violate Axiom 2. Hence $\rho(x, z) = 1/2$ as desired.

If $\rho(x, y) = 1/2$ and $\rho(y, z) > 1/2$, then we cannot have $\rho(x, z) < 1/2$, for then ρ would violate Axiom 1 on x, y, z. Moreover, we cannot have $\rho(x, z) = 1/2$, for then there would exist $w \in X$ with $\rho(y, w) \neq 1/2 \neq \rho(z, w)$ and ρ would violate Axiom 2 on w, x, y, z. So we must have the desired conclusion $\rho(x, z) > 1/2 = \min\{\rho(x, y), \rho(y, z)\}$. The same reasoning applies to the remaining case $\rho(x, y) > 1/2$ and $\rho(y, z) = 1/2$. $\square$

Lemma 3 (He and Natenzon (2024))**.** *ρ is moderately transitive iff it is a MUM.*

We could obtain a separate MUM representation in each galaxy X_i by applying Lemma 3. However, the representations for different galaxies may not be able to be 'stitched' together into a single extended representation of ρ on the entire universe X. In particular, if the representations we obtain for two different galaxies have different transformation functions F in equation (1), there may not exist a single transformation that accommodates behavior in both galaxies at the same time. Hence, before applying Lemma 3, we first show that we can create an auxiliary binary choice rule $\hat{\rho}$ that (i) replicates the behavior of the original ρ in each galaxy; and (ii) satisfies the full moderate transitivity postulate across the entire universe of alternatives.

Lemma 4. *Let ρ_1, ρ_2 be binary choice rules on X_1, X_2 respectively, and satisfying moderate transitivity. There exists a binary choice rule ρ that extends ρ_1 and ρ_2 to the union $X_1 \cup X_2$ and still satisfies moderate transitivity.*

Proof. Let $K_i = \max\{\rho_i(x, y) : x, y \in X_i\}$ for $i = 1, 2$ and let $K = \max\{K_1, K_2\}$ be the largest choice probability observed across both domains. The case where $K = 1/2$ is trivial, so we now consider the case $K > 1/2$. Define the binary choice rule ρ on $X_1 \cup X_2$ by

$$\rho(x, y) = \begin{cases} \rho_1(x, y), & \text{if } x, y \in X_1 \\ \rho_2(x, y), & \text{if } x, y \in X_2 \\ K, & \text{if } x \in X_1 \text{ and } y \in X_2 \\ 1 - K, & \text{if } x \in X_2 \text{ and } y \in X_1 \end{cases}$$

and now we verify that ρ satisfies moderate transitivity. Let $\rho(x, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$. When $x, y, z \in X_1$ or $x, y, z \in X_2$ the desired conclusion follows from the fact that ρ_1, ρ_2 satisfy moderate transitivity.

In the remaining cases, by the definition of ρ and $K > 1/2$ we must have that $x \in X_1$ while $z \in X_2$. Then, we have $\rho(x, z) = K \geq \max\{\rho(x, y), \rho(y, z)\}$ and moderate transitivity holds. □

By induction, Lemma 4 implies there exists an auxiliary binary choice rule $\hat{\rho}$ on $X = X_1 \cup X_2 \cup \cdots \cup X_k$ which is identical to the original choice rule ρ on each galaxy X_i while preserving moderate transitivity across the entire universe.

Applying Lemma 3 the auxiliary $\hat{\rho}$ admits a MUM representation (1) with a utility u, a transformation F and a finite distance metric $\hat{d}$. Let d be an extended distance metric given by

$$d(x,y) = \begin{cases} \hat{d}(x,y), & \text{if } x, y \in X_i \text{ for some } i \\ \infty, & \text{if } x \in X_i \text{ and } y \in X_j \text{ with } i \neq j \end{cases}$$

To see that the generalized MUM with parameters u, F, d represents ρ, take any $x, y \in X$. If x, y are in the same galaxy, $d(x,y) = \hat{d}(x,y)$ and $\hat{\rho}(x,y) = \rho(x,y)$, hence the representation yields the desired $\rho(x,y)$. If x, y are in different galaxies, by part (i) of the definition of $\sim$ it must be that $\rho(x,y) = 1/2$ and since $d(x,y) = \infty$, the representation again yields the correct $\rho(x,y)$. $\square$

References

Aumann, Robert J, "Utility theory without the completeness axiom," *Econometrica: Journal of the Econometric Society*, 1962, pp. 445–462.

Bewley, Truman F, "Knightian decision theory: Part I," Discussion Paper, Cowles Foundation for Research in Economics, Yale University, New Haven, CT 1986.

_ , "Knightian decision theory. Part I," *Decisions in economics and finance*, 2002, *25*, 79–110.

Bock, Robert Darrell and Lyle Vernon Jones, "Studies in the Measurement of Subjective Preference," *Psychometrika*, 1958, *23* (4), 309–323.

Cettolin, Elena and Arno Riedl, "Revealed preferences under uncertainty: Incomplete preferences and preferences for randomization," *Journal of Economic Theory*, 2019, *181*, 547–585.

Chambers, Robert G, Tigran Melkonyan, and John Quiggin, "Incomplete preferences, willingness to pay, and willingness to accept," *Economic Theory*, 2022, *74* (3), 727–761.

Chang, Ruth, *Incommensurability, Incomparability, and Practical Reason*, Harvard University Press, 1997.

Cubitt, Robin, Daniel Navarro-Martinez, and Chris Starmer, "On preference imprecision," *Journal of Risk and Uncertainty*, 2015, *50* (1), 1–34.

Danan, Eric and Anthony Ziegelmeyer, "Are preferences complete? An experimental measurement of indecisiveness under risk," *Papers on Strategic Interaction 2006-01*, 2006.

Dubra, Juan, Fabio Maccheroni, and Efe A Ok, "Expected utility theory without the completeness axiom," *Journal of Economic Theory*, 2004, *115* (1), 118–133.

Eliaz, Kfir and Efe A Ok, "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences," *Games and economic behavior*, 2006, *56* (1), 61–86.

Faro, José Heleno, "Variational bewley preferences," *Journal of Economic Theory*, 2015, *157*, 699–729.

Fishburn, Peter C, "Binary choice probabilities: on the varieties of stochastic transitivity," *Journal of Mathematical Psychology*, 1973, *10* (4), 327–352.

Fudenberg, Drew, Ryota Iijima, and Tomasz Strzalecki, "Stochastic Choice and Revealed Perturbed Utility," *Econometrica*, November 2015, *83* (6), 2371–2409.

Galaabaatar, Tsogbadral and Edi Karni, "Subjective expected utility with incomplete preferences," *Econometrica*, 2013, *81* (1), 255–284.

Halevy, Yoram, David Walker-Jones, and Lanny Zrill, "Difficult decisions," *Working Paper*, 2023.

Halff, Henry M., "Choice theories for differentially comparable alternatives," *Journal of Mathematical Psychology*, 1976, *14* (3), 244–246.

He, Junnan and Paulo Natenzon, "Moderate Utility," *American Economic Review: Insights*, 2024, *6* (2), 176–195.

Hotelling, Harold, "Stability in Competition," *The Economic Journal*, 1929, *39* (153), 41–57.

Luce, R. Duncan, *Individual Choice Behavior: a Theoretical Analysis*, Wiley New York, 1959.

McFadden, Daniel, "Modelling the Choice of Residential Location," *Spatial Interaction Theory and Planning Models*, 1978, *673*, 75–96.

Nielsen, Kirby and Luca Rigotti, "Revealed Incomplete Preferences," *Working Paper*, 2024.

Ok, Efe A, "Utility representation of an incomplete preference relation," *Journal of Economic Theory*, 2002, *104* (2), 429–449.

_ **, Pietro Ortoleva, and Gil Riella**, "Incomplete preferences under uncertainty: Indecisiveness in beliefs versus tastes," *Econometrica*, 2012, *80* (4), 1791–1808.

Raz, Joseph, "Value Incommensurability: Some Preliminaries," *Proceedings of the Aristotelian Society*, 1985, *86*, 117–134.

_ , *The Morality of Freedom*, Oxford University Press, 1986.

Restle, Frank, *Psychology of Judgment and Choice: A Theoretical Essay*, New York: John Wiley & Sons, 1961.

Salop, Steven C., "Monopolistic Competition with Outside Goods," *The Bell Journal of Economics*, 1979, *10* (1), 141–156.

Tversky, Amos, "Elimination by Aspects: A Theory of Choice," *Psychological Review*, 1972, *79* (4), 281–299.

_ **and J. Edward Russo**, "Substitutability and similarity in binary choices," *Journal of Mathematical Psychology*, 1969, *6* (1), 1–12.